Quantum technology, group theory, phase space

Lecture 1, Peking University 2019

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Outline

1. Exponential complexity
2. Phase-space methods
3. Non-classical phase-space
4. Extended phase-space

Quantum technology

ULTRALOW temperatures down to $1nK$

TESTS QUANTUM THEORY IN NEW REGIMES!

- Quantum optics
- Bose-Einstein condensates: atom 'photons'
- Quantum superfluid fermions: atom 'electrons'
- Optomechanics: interacting photons and phonons
- Quantum circuits: microwave entanglement, QC
- EARLY UNIVERSE?

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Traditional quantum theory methods?

- numerical diagonalisation?
  intractable for $\gtrsim 5$ particles
- operator factorization
  not applicable for strong correlations
- perturbation theory
  diverges at strong couplings
- exact solutions
  not applicable for quantum dynamics

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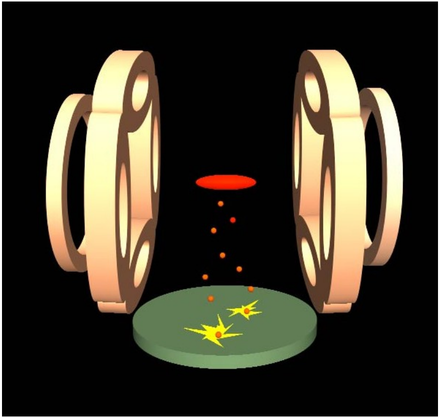
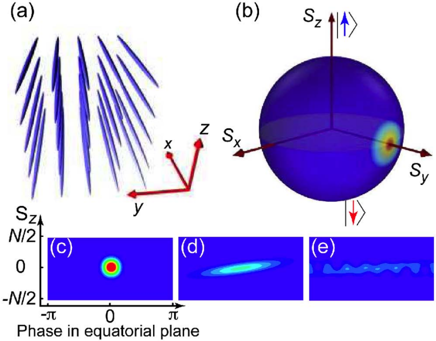
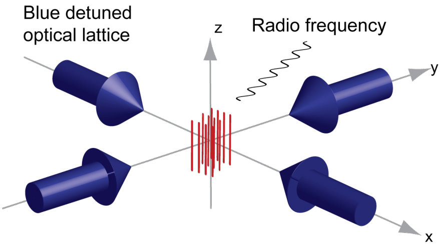
Atom correlation experiments



Quantum transport and interferometry experiments





General Hamiltonian with cold atoms

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_{INT} .$$

where $\widehat{H}_0$ and $\widehat{H}_{INT}$ are the non-interacting and interacting parts of the Hamiltonian respectively, so that H_0 is a general linear Hamiltonian.:

$$\widehat{H}_0 = \int \sum_{ss'} \left[\frac{\hbar^2}{2m_s} \nabla \widehat{\Psi}^\dagger_{s\mathbf{r}} \nabla \widehat{\Psi}_{s\mathbf{r}} \delta_{ss'} + \widehat{\Psi}^\dagger_{s\mathbf{r}} V_{ss'\mathbf{r}} \widehat{\Psi}_{s'\mathbf{r}} \right] d^3\mathrm{r} .$$

and $\widehat{H}_{INT}$ describes particle-particle interactions:

$$\widehat{H}_{INT} = \frac{1}{2} \sum_{ss'} \int \int \widehat{\Psi}^\dagger_{s\mathbf{r}} \widehat{\Psi}^\dagger_{s'\mathbf{r}'} U_{s\mathbf{r}s'\mathbf{r}'} \widehat{\Psi}_{s'\mathbf{r}'} \widehat{\Psi}_{s\mathbf{r}} d^3\mathbf{r} d^3\mathbf{r}' .$$

Local Mode Operators

Assume that the annihilation and creation operators are localized on a lattice $\{s_k, \mathbf{r}_k\}$ of species amd position indices, with lattive volume ΔV, so that:

$$\hat{a}^i = \sqrt{\Delta V}\widehat{\Psi}_{s_k \mathbf{r}_k}$$

In the case of bosonic (fermionic) fields, the commutators (anticommutators) are defined as:

$$\begin{aligned} \left\{\hat{a}^i, \hat{a}^{\dagger j}\right\}_{\pm} &= \delta^{ij} \\ \left\{\hat{a}^i, \hat{a}^j\right\}_{\pm} &= 0\,. \end{aligned}$$

The continuum Hamiltonian is regained in the limit of a large number of lattice sites:

$$\widehat{H}(\hat{\boldsymbol{a}}^{\dagger}, \hat{\boldsymbol{a}}) = \lim_{\Delta V \to 0} \hbar \left[\omega_{ij} \hat{a}^{\dagger i} \hat{a}^j + \frac{1}{2} \chi_{ij} : \widehat{n}^i \widehat{n}^j : \right].$$

Master equation

- $\hat{H} = \hat{H}_{sys} + \hat{H}_{sys-res} + \hat{H}_{res}$.
  - $\hat{H}_{sys}$ is the isolated system Hamiltonian
  - $\hat{H}_{sys-res} = \sum_j c_j \hat{A}_j \hat{R}_j^\dagger + hc$ is the coupling of the system to the reservoir
  - $\hat{H}_{res}$ is the reservoir Hamiltonian
- Γ_j is the amplitude decay rate of the system, caused by the reservoir
- The damping/gain operators $\hat{A}_j$ are system operators coupled to the reservoirs, with damping/gain constants Γ_j.

Master equation table

Damping operator $(\hat{A}_j)$	Γ_j	Physical interpretation
$\hat{a}_j$	γ_j	Linear (single-photon) decay
$\hat{a}_j^\dagger$	g_j	Linear (single-photon) gain
$\hat{a}_j^\dagger \hat{a}_j$	γ_j^p	Phase decay
$\hat{a}_j^2$	κ_j	Nonlinear (two-photon) decay

- All these terms typically occur in real quantum technology experiments
- Also have more complicated coupled-reservoir and high-order damping

Time evolution in position space

- The density matrix $\hat{\rho}$ evolves as:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{sys}, \hat{\rho}\right] + \sum_j \Gamma_j \mathscr{L}_j[\hat{\rho}]$$

- Here the Liouville terms are derived from the Golden Rule, and describe coupling to the reservoirs:

$$\mathscr{L}_j[\hat{\rho}] = 2\hat{O}_j\hat{\rho}\hat{O}_j^\dagger - \hat{O}_j^\dagger\hat{O}_j\hat{\rho} - \hat{\rho}\hat{O}_j^\dagger\hat{O}_j$$

- For n-particle collisions: $\hat{O}_i = [\hat{a}_i(\mathbf{r})]^n$

Exponential complexity

- consider N particles distributed among M modes
  - Number of quantum states:

$$D = \frac{(M+N-1)!}{(M-1)!N!}$$

  - take $N \simeq M \simeq 500,000$: $D = 2^{2M} = 2^{1,000,000}$
- More linear equations than atoms in the universe
- Can't diagonalize $2^{1,000,000} \times 2^{1,000,000}$ matrix!

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Wigner: Nobel prize in physics, 1963

1963 Nobel Prize in Physics

- one half to **Eugene Paul Wigner**,
  - ***for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles***
- half jointly to Maria Goeppert Mayer and J. Hans D. Jensen
  - ***for their discoveries concerning nuclear shell structure***

Implicit and explicit phase-space mappings

There are many phase-space methods with the implicit definition

$$\widehat{\rho} = \int P(\alpha)\hat{\Lambda}_P(\alpha)d^2\alpha$$

or the explicit definition

$$Q(\alpha) = Tr\left[\hat{\rho}\hat{\Lambda}_Q(\alpha)\right]$$

Properties of Wigner/Moyal phase-space

- Maps quantum states into **classical phase-space** $\alpha = p + ix$
- **Advantage**: complexity grows linearly with number of modes!

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Moyal arriving in Australia

Classical phase-space time-evolution

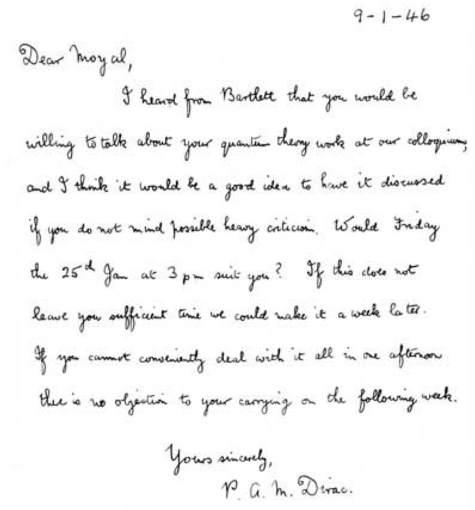
Moyal showed how to calculate time-evolution!

- Moyal brackets map quantum operators to differential equations
- **Famous correspondence with Dirac (who initially prevented publication)**
- Widely used in many areas of physics and elsewhere

Problem for computers: Distributions can have negative values

- **Later work of Husimi, Glauber, Sudarshan, Agarwal, Lax.**

Classical phase-space time-evolution

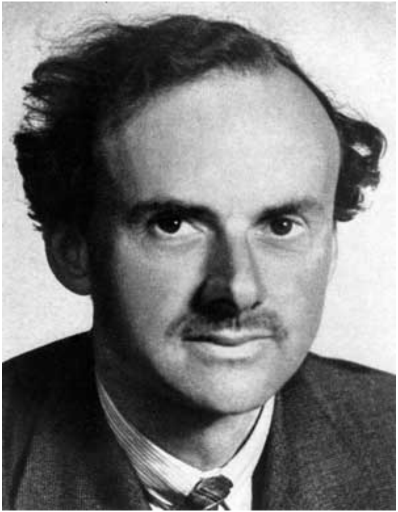


9-1-46

Dear Moyal,

I heard from Bartlett that you would be willing to talk about your quantum theory work at our colloquium, and I think it would be a good idea to have it discussed if you do not mind possible heavy criticism. Would Friday the 25th Jan at 3 pm suit you? If this does not leave you sufficient time we could make it a week later. If you cannot conveniently deal with it all in one afternoon there is no objection to your carrying on the following week.

Yours sincerely,
P. A. M. Dirac.



Wigner phase-space time-evolution

Moyal showed how to calculate time-evolution!

- Moyal: maps quantum operators to differential equations
- Moyal showed the theory was fully equivalent to quantum mechanics
- **Famous correspondence with Dirac**

Problem for computers: Wigner distributions have negative values

- **Gives a third-order Fokker-Planck equation**
- **No stochastic equivalent, unless truncated**
- **UV divergence in 3D**

Wigner-Moyal phase-space

$$\widehat{\rho} = \int W(\alpha)\hat{\Lambda}(\alpha)\, d^2\alpha$$

Generates symmetrically-ordered operator products

- Maps quantum states into 2M real coordinates: $\alpha = p + ix$,
- Advantage: Closest to classical behavior
- Problem: Nonpositive, UV divergent

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Operator identities

Differentiating the W projection operator gives the following four identities:

$$
\begin{aligned}
\widehat{a}_n^\dagger \widehat{\Lambda} &= \left[\frac{1}{2}\frac{\partial}{\partial \alpha_n} + \alpha_n^*\right]\widehat{\Lambda} \\
\widehat{a}_n \widehat{\Lambda} &= \left[-\frac{1}{2}\frac{\partial}{\partial \alpha_n^*} + \alpha_n\right]\alpha_n\widehat{\Lambda} \\
\widehat{\Lambda}\widehat{a}_n &= \left[\frac{1}{2}\frac{\partial}{\partial \alpha_n^*} + \alpha_n\right]\widehat{\Lambda} \\
\widehat{\Lambda}\widehat{a}_n^\dagger &= \left[-\frac{1}{2}\frac{\partial}{\partial \alpha_n} + \alpha_n^*\right]\widehat{\Lambda}
\end{aligned}
$$

Detailed equivalence

- Mapping of characteristic functions

$$W(\boldsymbol{\alpha}) = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z} \left\langle e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^*\cdot\left(\hat{\mathbf{a}}^\dagger-\boldsymbol{\alpha}^*\right)} \right\rangle$$

- Operator mean values
- $\left\langle \hat{a}_i^{\dagger m} \hat{a}_j^n \right\rangle_{SYM} = \int d^{2M}\boldsymbol{\alpha}\, \alpha_i^{*m} \alpha_j^n W(\boldsymbol{\alpha}) = \left\langle \alpha_i^{*m} \alpha_j^n \right\rangle_W$
- $\langle \hat{a}_j \rangle = \langle \alpha_j \rangle_W$
- $\left\langle \hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right\rangle /2 = \langle \alpha_i^* \alpha_j \rangle_W$

Dynamical mappings

Mapping of dynamical equations

$$\frac{\partial W(\boldsymbol{\alpha})}{\partial t} = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z}\, Tr\left[\frac{\partial \hat{\rho}}{\partial t} e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^{*}\cdot\left(\hat{\mathbf{a}}^{\dagger}-\boldsymbol{\alpha}^{*}\right)}\right]$$

Operator mappings

- $\hat{a}_j\hat{\rho} \to \left(\alpha_j + \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right) W$
- $\hat{\rho}\,\hat{a}_j^{\dagger} \to \left(\alpha_j^* + \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
- $\hat{a}_j^{\dagger}\hat{\rho} \to \left(\alpha_j^* - \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
- $\hat{\rho}\,\hat{a}_j \to \left(\alpha_j - \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right) W$

Example: Wigner function for a coherent state

Suppose we have a single-mode BEC in a coherent state

$$\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$$

Hence:

$$W(\alpha) = \frac{1}{\pi^2}\int d^2z\,\langle\alpha_0|\,e^{iz\cdot(\hat{a}-\alpha)+iz\cdot\left(\hat{a}^\dagger-\alpha^*\right)}\,|\alpha_0\rangle$$

Solution with a little algebra

- $$W(\alpha) = \frac{2}{\pi}e^{-2|\alpha-\alpha_0|^2}$$
  - **Exercise:** show that this gives $\langle\alpha^*\alpha\rangle = 1/2$ for a vacuum state

Example: time-evolution of harmonic oscillator

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -i\omega \left[\hat{a}^\dagger \hat{a} \widehat{\rho} - \widehat{\rho} \hat{a}^\dagger \hat{a}\right]$$

Operator mappings

- $\hat{a}^\dagger \hat{a} \hat{\rho} \to \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right)\left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W$
- $\hat{\rho} \hat{a}^\dagger \hat{a} \to \left(\alpha - \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right)\left(\alpha^* + \frac{1}{2}\frac{\partial}{\partial \alpha}\right) W$
- $$\frac{\partial W}{\partial t} = i\omega \left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right) W$$

Harmonic oscillator solution

Solution by method of characteristics

- $$\frac{\partial \alpha}{\partial t} = -i\omega\alpha$$
- $$\alpha(t) = \alpha(0)e^{-i\omega t}$$
- **Exercise:** Prove this!

Fokker-Planck equations

General result of operator mappings:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} + \frac{1}{6} \frac{\partial^3}{\partial \alpha_i \partial \alpha_j^* \partial \alpha_k^*} T_{ijk} + \dots \right\} W$$

Scaling to eliminate higher-order terms

$$x = \alpha / \sqrt{N}$$

$$\frac{\partial W}{\partial t} = \left\{ -\frac{1}{\sqrt{N}} \frac{\partial}{\partial x_i} A_i + \frac{1}{2N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} + \mathrm{O}\left(\frac{1}{N^{3/2}} \right) \right\} W$$

Stochastic equation

Result of operator mappings + truncation - valid if $N/M >> 1$:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} \right\} W$$

Equivalent stochastic equation

$$\frac{\partial \alpha_i}{\partial t} = A_i + \zeta_i(t)$$

where:

$$\left\langle \zeta_i(t) \zeta_j^*(t) \right\rangle = D_{ij} \delta \left(t - t' \right)$$

Example: BEC case

Result of operator mappings + truncation - for the GPE:

- $$\frac{d\psi_j(\mathbf{x})}{dt} = \left[iK_j\psi_j - i\int U_{ij}|\psi_i(\mathbf{x}')|^2 d\mathbf{x}' - \gamma_j \right] \psi_j(\mathbf{x}) + \sqrt{\gamma_j}\zeta_j(\mathbf{x},t)$$

  the linear unitary evolution of the wave-function, is:

  $$K_j = \hbar\nabla^2/2m - V_j(\mathbf{r})$$

  while $\zeta_i(\mathbf{x},t)$ is a complex, stochastic Gaussian noise:

  $$\left\langle \zeta_i(\mathbf{x},t)\zeta_j^*(\mathbf{x}',t') \right\rangle = \delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \,.$$

- Initial fluctuations: $\langle \Delta\Psi_s(\mathbf{x})\Delta\Psi_u^*(\mathbf{x}')\rangle = \frac{1}{2}\delta_{su}\delta^3(\mathbf{x}-\mathbf{x}')$

Phase-space representation methods have many applications

- Maps **quantum field evolution** into a stochastic equation
- Can also be used to treat interferometry
- Advantage: No exponential complexity issues!
- Mathematical challenge with Wigner method:
- truncation error needs to be checked

Coherence theory, lasers and phase-space

2005 Nobel Prize in Physics

- one half to **R. J. Glauber**
  - ***for his contribution to the quantum theory of optical coherence***
- one half to **T. Haensch** and **J. Hall**
  - ***for their contributions to the development of laser-based precision spectroscopy***

Glauber-Sudarshan Phase-space

$$\widehat{\rho} = \int P(\alpha) |\alpha\rangle \langle\alpha| d^2\alpha$$

Generates normal-ordered operator products

- Maps quantum states into 2M real coordinates: $\alpha = p + ix$,
- Advantage: No UV vacuum divergence
- Problem: Singular for entangled states

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+P PHASE-SPACE METHODS

$$\widehat{\rho} = \int P(\alpha,\beta)\frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*||\alpha\rangle}d^2\alpha d^2\beta$$

Enlarged phase-space allows positive probabilities!

- Maps quantum states into 4M real coordinates: $\alpha,\beta = p+ix, p'+ix'$
- Double the size of a classical phase-space
- Advantage: Can represent entangled states

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+P Existence Theorem

For ANY density matrix, a positive P-function always exists

$$P(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{4\pi^2} e^{-\left|\boldsymbol{\alpha}-\boldsymbol{\beta}^*\right|^2/4} \left\langle \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \middle| \widehat{\rho} \middle| \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2} \right\rangle$$

Enlarged phase-space allows positive probabilities!

- Advantage: Probabilistic sampling is possible
- Problem: Non-uniqueness allows sampling error to grows with time (chaotic)

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Differentiating the +P projection operator gives the following four identities:

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\begin{aligned}
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\widehat{\Lambda}\widehat{a}_n^{\dagger} &= \beta_n\widehat{\Lambda}
\end{aligned}
$$

Since the projector is an analytic function of both α_n and β_n, we can obtain alternate identities by replacing $\partial/\partial\alpha$ by either $\partial/\partial\alpha_x$ or $\partial/i\partial\alpha_y$. This equivalence allows a positive-definite diffusion to be obtained, with stochastic evolution.

Measurements

An important application of these identities is the property of measurement. In order to calculate an operator expectation value, there is a direct correspondence between the moments of the distribution, and the normally ordered operator products. These come directly from the fact that coherent state are eigenstates of the annihilation operator, and that $\mathrm{Tr}\left[\widehat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})\right]=1$:

$$\langle\widehat{a}_m^\dagger\cdots\widehat{a}_n\rangle=\int\int P(\boldsymbol{\alpha},\boldsymbol{\beta})[\beta_m\cdots\alpha_n]d^{2M}\boldsymbol{\alpha}\,d^{2M}\boldsymbol{\beta}\,.$$

Weighted stochastic gauge equations

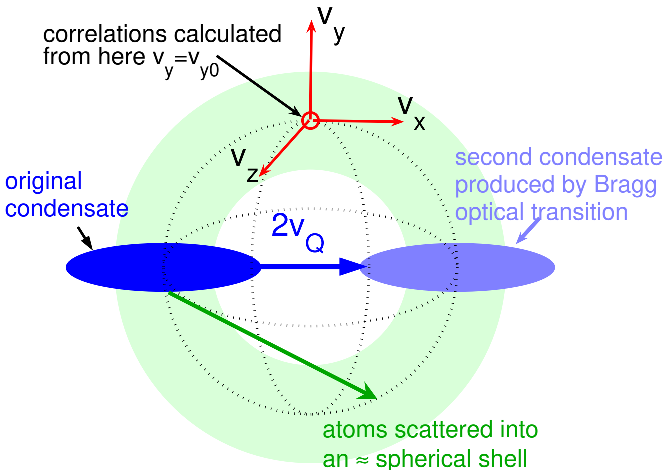
Exponential quantum problems $\rightarrow$ stochastic equations, eg single-component Bose gas, S-wave interactions:

$$i\hbar\frac{d\alpha_{\vec{x}}}{dt} = \left[-\frac{\hbar^2}{2m}\nabla^2 + U\alpha_{\vec{x}}\beta_{\vec{x}} + \sqrt{i\hbar g}\ \xi_{\vec{x}}\right]\alpha_{\vec{x}}$$

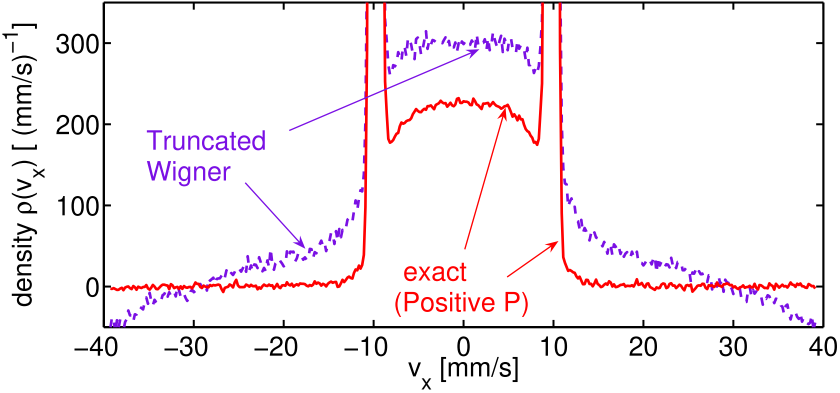
$$-i\hbar\frac{d\beta_{\vec{x}}}{dt} = \left[-\frac{\hbar^2}{2m}\nabla^2 + U\alpha_{\vec{x}}\beta_{\vec{x}} + \sqrt{-i\hbar g}\ \xi^+_{\vec{x}}\right]\beta_{\vec{x}}$$

- **Can be used for bosons and fermions (with modifications)**
- Many trajectories needed to control growing sampling errors
- Choice of basis, 'gauge' and stochastic method is possible

BEC collision: 10^5 bosons, 10^6 spatial modes



Positive-P vs Truncated Wigner



3D Truncated Wigner: diverges, +P: converges

+P advantages and drawbacks

- Advantage: Can treat exponentially large systems
- First-principles approach WITHOUT factorization assumption
- No truncation
- No UV divergence at large k-value
- Drawback: Sampling error grows in time
- Can't simulate unitary evolution for long times!

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General phase-space approach

$$\widehat{\rho} = \int P(\overrightarrow{\lambda})\widehat{\Lambda}_P(\overrightarrow{\lambda})d\overrightarrow{\lambda}$$

$$Q(\overrightarrow{\lambda}) = Tr\left(\widehat{\Lambda}_Q(\overrightarrow{\lambda})\hat{\rho}\right)$$

Phase-space may be larger still!

- Here $\widehat{\Lambda}(\overrightarrow{\lambda})$ must be complete
- Quantum dynamics $\rightarrow$Trajectories in $\overrightarrow{\lambda}$.
- Different basis choice $\widehat{\Lambda}(\overrightarrow{\lambda}) \rightarrow$ different representation
- Eg, positive P-representation: $\widehat{\Lambda}(\overrightarrow{\lambda}) = |\alpha\rangle\langle\beta| / \langle\beta||\alpha\rangle$

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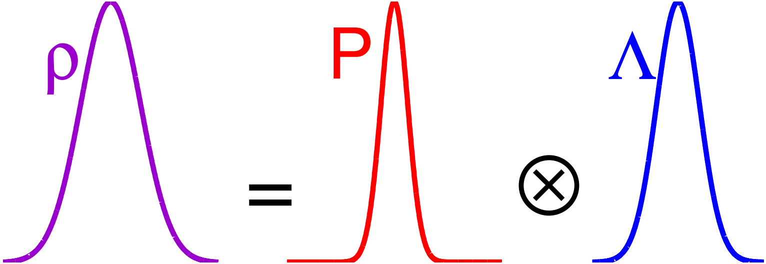
Trade-offs: distribution vs basis



$$\sigma_\rho \sim \sigma_P + \sigma_\Lambda$$

General M-mode Gaussian operator

Normally-ordered exponential of a quadratic form in the $2M$-vector mode operator $\delta\underline{\widehat{a}} = (\widehat{\mathbf{a}}, \widehat{\mathbf{a}}^{\dagger}) - \underline{\alpha}$, where $\underline{\alpha}$ is a c-vector and $\widehat{\mathbf{a}}$ is the vector of annihilation operators. Used for either bosons or fermions:

$$\widehat{\Lambda}(\overrightarrow{\lambda}) = \frac{\Omega}{\sqrt{\left|\underline{\underline{\sigma}}\right|}} : \exp\left[-\delta\underline{\widehat{a}}^{\dagger}\underline{\underline{\sigma}}^{-1}\delta\underline{\widehat{a}}/2\right] : \; .$$

Quantum phase-space: $\overrightarrow{\lambda} = (\Omega, \underline{\alpha}, \underline{\underline{\sigma}})$.

What is the covariance?

$$\underline{\underline{\sigma}} = \begin{bmatrix} \mathbf{I}+\mathbf{n} & \mathbf{m} \\ \mathbf{m}^{+} & \mathbf{I}+\mathbf{n}^{T} \end{bmatrix}.$$

Eg, fermion case: representation phase space is $\vec{\lambda} = (\Omega, \mathbf{n}, \mathbf{m}, \mathbf{m}^{+})$

- Ω= weight factor
- $\mathbf{n}$ = number correlation - OBSERVABLE
- $\mathbf{m}, \mathbf{m}^{+}$= anomalous correlation - OBSERVABLE

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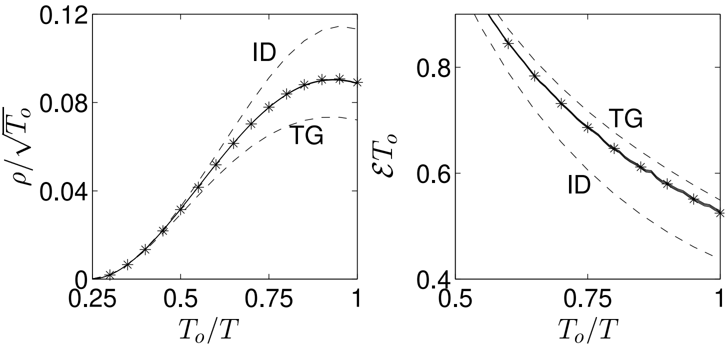
Weighted stochastic gauge equations

Exponential quantum problems → tractable stochastic equations

$$d\Omega/\partial t = \Omega\left[U + \mathbf{g}\cdot\boldsymbol{\zeta}\right]$$
$$d\boldsymbol{\alpha}/\partial t = \mathbf{A} + \mathbf{B}(\boldsymbol{\zeta} - \mathbf{g})$$

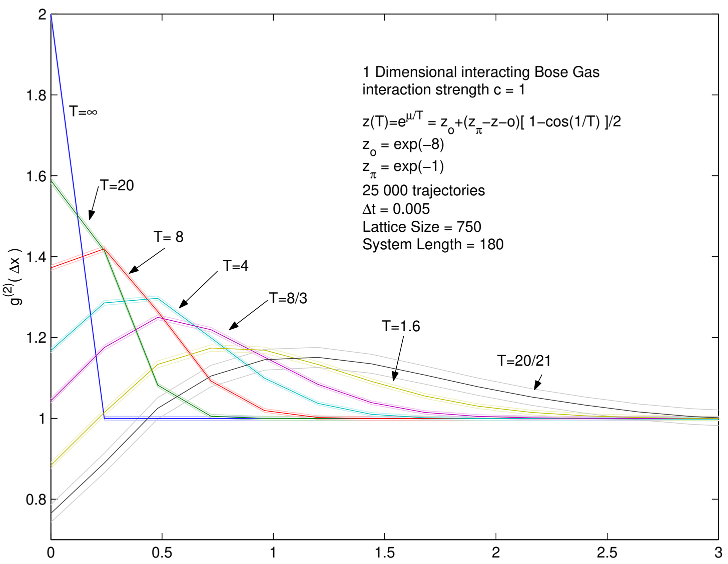
- **Can be used for fermions AND bosons**
- Many trajectories needed to control growing sampling errors
- **g is a gauge chosen to stabilize trajectories**
- A choice of basis, gauge and stochastic method is necessary

ONE-DIMENSIONAL BEC



Agreement of simulations with exact solutions

Predicts: anomalous spatial correlations



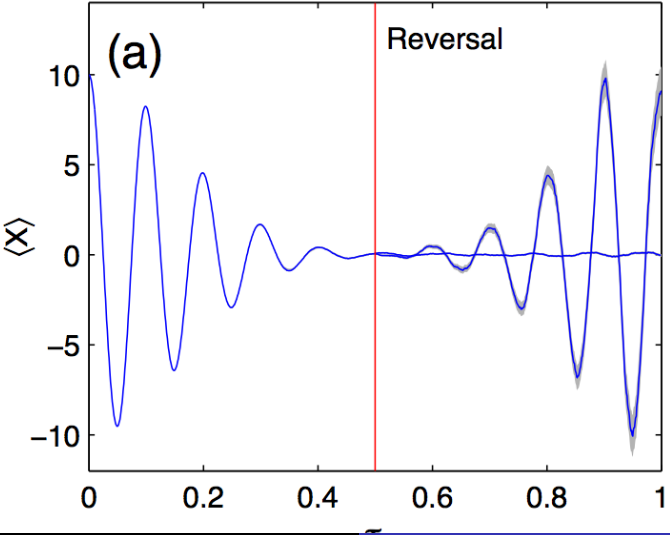
BEC QUANTUM DYNAMICS

Single-mode case:

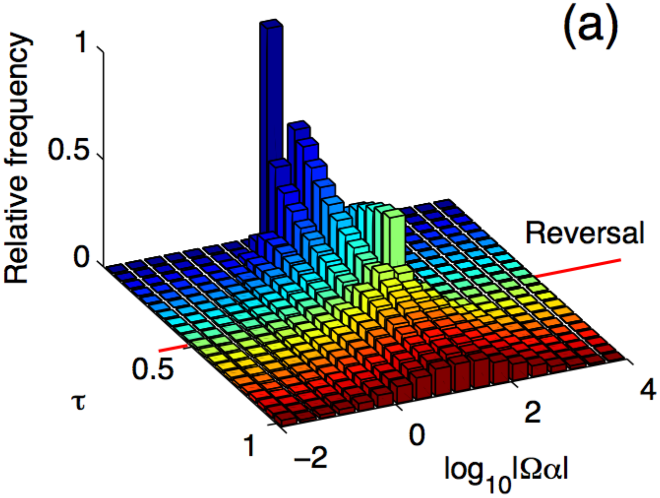
$$
\begin{aligned}
i\frac{d\alpha}{d\tau} &= \left[|\alpha\beta^*| + \omega + \sqrt{i}\zeta_1(\tau)\right]\alpha \\
i\frac{d\beta}{d\tau} &= \left[|\alpha\beta^*| + \omega + \sqrt{i}\zeta_2(\tau)\right]\beta \\
\frac{d\Omega}{d\tau} &= \Omega g_i \zeta_i(\tau)
\end{aligned}
$$

- Unitary evolution of 10^{23} interacting bosons

Time-reversal test: up to 10^{23} interacting bosons



Phase-space distribution is not unique!



SUMMARY

Phase-space representation methods have many applications

Enlarged phase-space makes them more powerful!

- Maps **quantum field evolution** into a stochastic equation
- Can also be used to treat genetics and population dynamics
- Advantage: No exponential complexity issues!
- Mathematical challenge:
  - sampling error often increases with time,
  - basis needs careful choice

SUMMARY

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